Hypersonic Viscous Flow Over a Sweat-Cooled Flat Plate

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Summary

This paper presents a theoretical analysis of the hypersonic viscous flow over a sweat-cooled flat plate. The physical system under consideration is the hypersonic laminar boundary layer over a porous flat plate with homogeneous, normal injection of a coolant into the external stream. A heat balance at the porous surface is made between the heat transferred to the surface and the heat absorbed by the coolant. The existence of similar solutions requires a nonuniform distribution of coolant injection. The method of solution consists of the integration of three simultaneous first-order equations, the momentum and the energy integral equations in the boundary layer, and the tangent-wedge approximation in the inviscid layer. First-order asymptotic formulas are given in both the strong and the weak pressure interaction regions for the induced surface pressure, the skin-friction coefficient, and the Nusselt number. Numerical results for three specific cases are presented and discussed.

Symbols

A = function defined by Eq. (20a)
a_n = coefficient of η^n in the velocity profile
b_n = coefficient of η^n in the stagnation-enthalpy profile
C_∞ = viscosity-temperature proportionality constant, defined by Eq. (4)
C_f = local skin-friction coefficient, defined by Eq. (32)
c_p = specific heat of gas at constant pressure
E = function defined by Eq. (10b)
F = function defined by Eq. (10a)
H = stagnation-enthalpy, defined as $[(1/2)u^2 + C_pT]$
h = (H_0/H_1)
I = function defined by Eq. (20b)
i_c, i_1 = values defined by Eq. (26c)
J = function defined by Eq. (10c)
j_0, j_1 = values defined by Eq. (26b)
k = thermal conductivity of gas
M = Mach number
Nu = local Nusselt number, defined by Eq. (35)
p = pressure of gas
T = absolute temperature
T_c = temperature of the coolant
T_e = equilibrium insulated wall temperature
t = transformation variable defined by Eq. (6a)
u = velocity of flow along the flat plate
v = velocity of flow normal to the flat plate
$\bar{v}$ = variable defined by Eq. (6b)
x = distance along the flat plate
y = distance normal to the flat plate
γ = ratio of specific heats
Δ = dimensionless boundary-layer thickness, defined by Eq. (25b)
δ = boundary-layer thickness in (x, y) plane
$\bar{\delta}$ = boundary-layer thickness in (x, t) plane
η = variable defined by Eq. (10d)
Λ = Pohlhausen parameter, defined by Eq. (13)
μ = coefficient of viscosity
ν = kinematic viscosity
ρ = density of gas
Φ = injection parameter, defined by Eq. (10e)
$\bar{\chi}$ = interaction parameter, defined by Eq. (25a)

Subscripts

0 = values at wall (exceptions, a_0, i_0, and j_0)
1 = values at outer edge of boundary layer (exceptions, a_1, i_1, and j_1)
∞ = values at an undisturbed region of the flow

Superscripts

(0) = zero-order coefficients in the expansion in $\bar{\chi}$
(1) = first-order coefficients in the expansion in $\bar{\chi}$

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Introduction

THE HYPERSONIC VISCOUS FLOW over a flat plate with surface mass transfer has been studied in the region of strong pressure interactions by Yasuhara[1] for the insulated flat plate, and by Li and Gross[2] for both the insulated and the noninsulated flat plates. By application of a method originally proposed by Nagakura and Naruse[3] for the treatment of the hypersonic viscous flow over an impervious plate, Tien[4] has extended the treatment to the case with surface mass transfer. He obtained results in both the strong and the weak pressure interaction regions for an insulated plate. His results for the zero-order strong interactions agree favorably with the results of Li and Gross. The case for the insulated flat plate, however, is not as physically important as that for the noninsulated plate. In most cases, the main purpose of having a surface mass transfer is to protect surfaces from hot gas streams, such as the surfaces of high-speed vehicles. The problem of practical interest is that of maintaining a certain uniform surface temperature by injecting a coolant gas into the hot gas stream.

The purpose of the present work is to study the effects of coolant injection on both the flow and the heat-transfer characteristics in the hypersonic stream over a porous flat plate. The coolant gas in this investigation is assumed to be the same as the gas in the main stream, and the coolant injection velocity is normal to the surface. The required injection velocity will be, in general, a function of the coolant temperature, the desired uniform surface temperature, the free-stream Mach number, and the distance from the leading edge. The variation of the injection velocity along the plate surface is determined by the requirement for the existence of similar solutions. Approximate expressions of the induced surface pressure, the boundary-layer thickness, the skin-friction coefficient, and the Nusselt number will also be given for both the regions of strong and weak pressure interactions.

The present physical problem is different from that considered by Li and Gross for the noninsulated flat plate in that a heat balance at the porous surface has been considered here. At steady state, this heat balance is characterized by an equation stating that the heat transferred by the hot gas to the wall is absorbed by the coolant flow in the wall. This type of heat balance has been considered by several authors in treating sweat-cooling problems.[5] The wall temperature is thus related to the rate of mass injection instead of being independent of each other as in the case considered by Li and Gross. Therefore, the emphasis in the present analysis is placed upon the effects of a coolant injection on the hypersonic viscous interactions.

The treatment in the present study is an extension of the method of Nagakura and Naruse[3] to the sweat-cooling problems in hypersonic flow. The Howarth-Dorodnitsyn transformation is first applied to the equations of motion. By using one boundary-layer thickness and applying the Kármán-Pohlhausen integral method, the momentum and the energy boundary-layer equations, combined with the heat-balance equation at the wall, are simplified and integrated. The approximate final results are then obtained by solving simultaneously the integrated boundary-layer equations and the tangent-wedge expression in the inviscid region. The less accurate tangent-wedge approximation is used here in view of its simplicity in the present treatment. The final solution, however, is expected to give reasonably approximate results, as good agreement has been established in several cases[3, 4, 6] between the results from the present approach and those from other approaches.

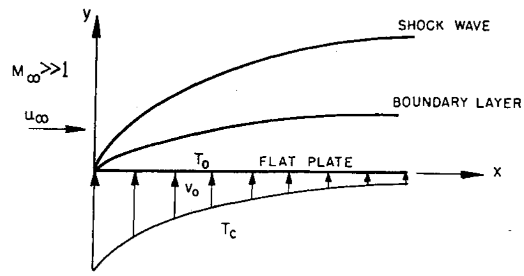

FIG. 1. Diagram of the physical system.

Analysis

The semi-infinite flat plate is placed at zero angle of attack with respect to a uniform hypersonic stream. A shock wave extends back from the leading edge, which is located at $x = 0$ and $y = 0$. The region between the shock and the plate surface is divided into two regions, the inviscid region, and the boundary layer. A schematic diagram of the physical system is shown in Fig. 1. The main portion of this section will be the development of the integrated boundary-layer equations.

The governing equations for the boundary-layer region, under the assumption that the fluid is a perfect gas of constant specific heats and Prandtl number of unity, may be written as

$$(\partial \rho u/\partial x) + (\partial \rho v/\partial y) = 0 \tag{1a}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{1b}$$

$$\partial p/\partial y = 0 \tag{1c}$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial H}{\partial y}\right) \tag{1d}$$

$$p/p_\infty = \rho T/\rho_\infty T_\infty \tag{1e}$$

The coolant, which is assumed to be the same as the fluid in the main stream, is injected normal to the plate surface. A steady heat-balance equation can then be established at the porous surface by equating the heat transferred from the hot fluid to the wall and the heat absorbed by the coolant—i.e.,

$$k_0(\partial T/\partial y)_0 = \rho_0 v_0 C_p(T_0 - T_c) \tag{2}$$

where v_0 is to be much smaller than the velocity in the free stream,

$$v_0^2 \ll u_\infty^2 \tag{3}$$

With the assumption of a linear relationship between the viscosity and the temperature

$$(\mu/\mu_\infty) = C_\infty(T/T_\infty) \tag{4}$$

the governing equations can be expressed as

$$\frac{u}{\rho_0}\frac{\partial \rho_0}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial \bar{v}}{\partial t} = 0 \tag{5a}$$

$$u \frac{\partial u}{\partial x} + \bar{v} \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial x} + \nu_0 \frac{\partial^2 u}{\partial t^2} \tag{5b}$$

$$u \frac{\partial H}{\partial x} + \bar{v} \frac{\partial H}{\partial t} = \nu_0 \frac{\partial^2 H}{\partial t^2} \tag{5c}$$

where

$$t \equiv \int_0^y \left(\frac{\rho}{\rho_0}\right) dy \tag{6a}$$

and

$$\bar{v} \equiv u(\partial t/\partial x) + (\rho/\rho_0)v \tag{6b}$$

The above equations must satisfy the following boundary conditions:

at $t = 0$,

$$\left.\begin{aligned}
&\frac{u}{u_1} = 0, \quad \frac{H}{H_1} \equiv h, \quad \bar{v} = \bar{v}_0 \\
&v_0 u_1 \left[\frac{\partial}{\partial t}\left(\frac{u}{u_1}\right)\right]_0 = -\frac{1}{\rho}\frac{dp_1}{dx} + \nu_0 u_1 \left[\frac{\partial^2}{\partial t^2}\left(\frac{u}{u_1}\right)\right]_0 \\
&v_0 \left[\frac{\partial}{\partial t}\left(\frac{H}{H_1}\right)\right]_0 = \nu_0 \left[\frac{\partial^2}{\partial t^2}\left(\frac{H}{H_1}\right)\right]_0 \\
&\text{at } t = \bar{\delta}, \\
&\left(\frac{u}{u_1}\right) = 1, \quad \frac{\partial}{\partial t}\left(\frac{u}{u_1}\right) = \frac{\partial^2}{\partial t^2}\left(\frac{u}{u_1}\right) = 0 \\
&\left(\frac{H}{H_1}\right) = 1, \quad \frac{\partial}{\partial t}\left(\frac{H}{H_1}\right) = \frac{\partial^2}{\partial t^2}\left(\frac{H}{H_1}\right) = 0
\end{aligned}\right\} \tag{7}$$

where $\bar{\delta}$ according to Eq. (6a) is defined as

$$\bar{\delta} \equiv \int_0^{\delta} \left(\frac{\rho}{\rho_0}\right) dy \tag{8}$$

and δ is the boundary-layer thickness. The last two conditions at $t = 0$ are compatibility conditions at the surface and were cited above because of the integral treatment to be used in the later analysis.

Integration of Eqs. (5a)–(5c) with respect to t from $t = 0$ to $t = \bar{\delta}$, with the appropriate boundary conditions (7), yields the following equations:

$$\frac{d}{dx}(\bar{\delta}^2) + \bar{\delta}^2 \left(\frac{p_\infty}{p_1}\right)\frac{d}{dx}\left(\frac{p_1}{p_\infty}\right)\left[2 - \left(\frac{4F + 2E}{\gamma M_1^2 F}\right)\right] + \frac{2\bar{\delta}^2}{F}\frac{dF}{dx} = \frac{2\nu_0}{F u_1}\left\{\Phi + \left[\frac{\partial}{\partial \eta}\left(\frac{u}{u_1}\right)\right]_0\right\} \tag{9a}$$

and

$$\frac{d}{dx}(\bar{\delta}^2) + \bar{\delta}^2 \left(\frac{p_\infty}{p_1}\right)\frac{d}{dx}\left(\frac{p_1}{p_\infty}\right)\left[2 - \frac{2}{\gamma M_1^2}\right] + \frac{2\bar{\delta}^2}{J}\frac{dJ}{dx} = \frac{2\nu_0}{J u_1}\left\{\Phi(1 - h) + \left[\frac{\partial}{\partial \eta}\left(\frac{H}{H_1}\right)\right]_0\right\} \tag{9b}$$

where

$$F \equiv \int_0^1 \left(\frac{u}{u_1}\right)\left(1 - \frac{u}{u_1}\right) d\eta \tag{10a}$$

$$E \equiv \int_0^1 \left(\frac{T}{T_1} - \frac{u}{u_1}\right) d\eta \tag{10b}$$

$$J \equiv \int_0^1 \left(\frac{u}{u_1}\right)\left(1 - \frac{H}{H_1}\right) d\eta \tag{10c}$$

$$\eta \equiv t/\bar{\delta} \tag{10d}$$

and the injection parameter,

$$\Phi \equiv v_0 \bar{\delta}/\nu_0 \tag{10e}$$

Note that the integration of the momentum and the energy equations involves only one boundary-layer thickness. This, however, does not impose any undue limitations on the stagnation-enthalpy profiles, since these profiles are to be permitted to contain an additional unknown coefficient, as discussed in Ref. 7. This additional coefficient replaces the thermal boundary-layer thickness as a parameter. The use of a single boundary-layer thickness makes the analysis somewhat simpler. The velocity and stagnation-enthalpy profiles are thus given in the following polynomial forms:

$$\left(\frac{u}{u_1}\right) = \sum_{n=0}^{4} a_n \eta^n, \quad \left(\frac{H}{H_1}\right) = \sum_{n=0}^{5} b_n \eta^n \tag{11}$$

where the coefficients a_n and b_n, as determined from the boundary conditions (7), are

$$a_0 = 0, \quad a_1 = \frac{\Lambda + 12}{\Phi + 6}, \quad a_2 = \frac{6\Phi - 3\Lambda}{\Phi + 6}$$

$$a_3 = -\frac{(8\Phi + 12 - 3\Lambda)}{(\Phi + 6)}, \quad a_4 = \frac{(3\Phi + 6 - \Lambda)}{(\Phi + 6)} \tag{12a}$$

and

$$\begin{aligned}
&b_0 = h, \quad b_1 = b_1, \quad b_2 = \frac{1}{2}\Phi b_1 \\
&b_3 = 10 - 10h - 6b_1 - (3/2)\Phi b_1 \\
&b_4 = -15 + 15h + 8b_1 + (3/2)\Phi b_1 \\
&b_5 = 6 - 6h - 3b_1 - (1/2)\Phi b_1
\end{aligned} \tag{13b}$$

where Λ is the Pohlhausen parameter,

$$\Lambda \equiv -(\bar{\delta}^2/\mu_0\rho_1)(dp_1/dx) \tag{13}$$

The temperature in the boundary layer, from the definition of stagnation enthalpy and the assumption of small mass injection, can be expressed as:

$$\left(\frac{T}{T_1}\right) = -\left(\frac{\gamma - 1}{2}\right) M_1^2 \left(\frac{u}{u_1}\right)^2 + \left(\frac{H}{H_1}\right)\left[1 + \left(\frac{\gamma - 1}{2}\right) M_1^2\right] \tag{14}$$

Since the Mach number of the free stream is large, $M_\infty^2 \gg 1$, and consequently $M_1^2 \gg 1$, the temperature at the wall is given approximately as

$$(T_0/T_1) = h[(\gamma - 1)/2]M_1^2 \tag{15a}$$

and similarly,

$$(T_0/T_\infty) = h[(\gamma - 1)/2]M_\infty^2 \tag{15b}$$

These relations will be useful in simplifying the integral equations.

For a Prandtl number of unity the equilibrium insulated wall temperature is related to the temperature at the outer edge of the boundary layer

$$T_e = T_1\left[1 + \left(\frac{\gamma - 1}{2}\right) M_1^2\right] \simeq T_1\left(\frac{\gamma - 1}{2}\right) M_1^2 \tag{16}$$

and thus,

$$h \equiv H_0/H_1 = T_0/T_e \qquad (17)$$

The ratio, h, is less than one for a sweat-cooled surface.

The heat-balance Eq. (2), in terms of dimensionless parameters, can be written as:

$$b_1 = \Phi h[1 - (T_c/T_0)] \qquad (18)$$

With appropriate manipulations, and neglecting terms of order $O[M_1^{-2}]$, the momentum and the energy integral equations become respectively

$$\frac{d}{dx}(\bar{\delta}^2) + \bar{\delta}^2\left(\frac{p_\infty}{p_1}\right)\frac{d}{dx}\left(\frac{p_1}{p_\infty}\right)\left\{2 - \left(\frac{\gamma - 1}{\gamma}\right)\left[\frac{I(\Phi + 6) - h}{F(\Phi + 6)}\right]\right\} + \frac{2\bar{\delta}^2}{F}\frac{dF}{dx} = \frac{2AC_\infty\nu_\infty}{Fu_\infty}\left(h\,\frac{\gamma - 1}{2}\,M_\infty^2\right)^2\left(\frac{p_\infty}{p_1}\right) \qquad (19a)$$

and

$$\frac{d}{dx}(\bar{\delta}^2) + 2\bar{\delta}^2\left(\frac{p_\infty}{p_1}\right)\frac{d}{dx}\left(\frac{p_1}{p_\infty}\right) + \frac{2\bar{\delta}^2}{J}\frac{dJ}{dx} = \frac{2C_\infty\nu_\infty}{Ju_\infty}\left(h\,\frac{\gamma - 1}{2}\,M_\infty^2\right)^2\left(1 - h\,\frac{T_c}{T_0}\right)\Phi\left(\frac{p_\infty}{p_1}\right) \qquad (19b)$$

where

$$A \equiv \frac{(\Phi^2 + 6\Phi + 12)}{(\Phi + 6)} \qquad (20a)$$

$$I \equiv \int_0^1 \left[\frac{H}{H_1} - \left(\frac{u}{u_1}\right)^2\right] d\eta \qquad (20b)$$

It should be emphasized here that in writing the energy integral Eq. (19b) and also in carrying out the integration of this equation later, the function, J, has been assumed to be nonzero. From the definition of J, Eq. (10c), this means that results for the case of an insulated flat plate can not be obtained simply by putting into the final solution $h = 1$ and $T_c = T_0$, as required for an insulated surface from Eqs. (2) and (17). The case of an insulated plate with mass injection, however, is much simpler since the energy equation can be replaced by $h = 1$, and the solution of this case has been obtained by the same method as used here.[4]

The explicit expressions for F, J, and I can be obtained by use of the assumed velocity and stagnation-enthalpy profiles, Eqs. (11), and the heat-balance equation, Eq. (18). They are,

$$F = \frac{(21 + 3\Phi)}{5(\Phi + 6)} - \frac{1}{35(\Phi + 6)^2}(734 + 221\Phi + 17\Phi^2) + \frac{\Lambda}{20(\Phi + 6)} - \frac{\Lambda}{420(\Phi + 6)^2}(142 + 17\Phi) - \frac{\Lambda^2}{252\,(\Phi + 6)^2} \qquad (21a)$$

$$J = \frac{1}{(\Phi + 6)}\left\{-\left[\frac{51}{35}(h - 1) + \frac{79}{210}\Phi h\left(1 - \frac{T_c}{T_0}\right)\right] - \Phi\left[\frac{71}{420}(h - 1) + \frac{23}{280}\Phi h\left(1 - \frac{T_c}{T_0}\right)\right] - \frac{23}{5040}\Phi^3 h\left(1 - \frac{T_c}{T_0}\right) - \Lambda\left[\frac{31}{840}(h - 1) + \frac{19}{2520}\Phi h\left(1 - \frac{T_c}{T_0}\right) + \frac{1}{1680}\Phi^2 h\left(1 - \frac{T_c}{T_0}\right)\right]\right\} \qquad (21b)$$

$$I = \left[\frac{1}{2}(1 + h) + \frac{1}{10}\Phi h\left(1 - \frac{T_c}{T_0}\right) + \frac{1}{120}\Phi^2 h\left(1 - \frac{T_c}{T_0}\right)\right] - \frac{1}{(\Phi + 6)^2}\left[\frac{734}{35} + \frac{221}{35}\Phi + \frac{17}{35}\Phi^2\right] - \frac{\Lambda}{(\Phi + 6)^2}\left[\frac{71}{210} - \frac{17}{420}\Phi\right] - \frac{\Lambda^2}{252(\Phi + 6)^2} \qquad (21c)$$

Eqs. (19a) and (19b) represent the governing equations in the boundary layer. For the inviscid region, the following relation[8] provided by the tangent-wedge approximation for a flat plate will be used:

$$\frac{d}{dx}\left(\frac{I\bar{\delta}}{h}\right) = \frac{1}{M_\infty}\left[\frac{2}{\gamma(\gamma + 1)}\right]^{1/2} \times \left(\frac{p_1}{p_\infty} - 1\right)\left(\frac{p_1}{p_\infty} + \frac{\gamma - 1}{\gamma + 1}\right)^{-1/2} \qquad (22)$$

where the boundary-layer thickness in (x,η) plane is related to the actual boundary-layer thickness by the relation:

$$(\delta/\bar{\delta}) = I/h \qquad (23)$$

The solution of the physical problem is then reduced to integrating Eqs. (19a), (19b), and (22) simultaneously for the three dependent variables, $\bar{\delta}$, (p_1/p_∞) and Φ with x as the independent variable. The expressions of F, J, and I are quite complicated; however, simplifications can be made from order-of-magnitude considerations. An approximate solution for the asymptotic behaviors in the weak and the strong interaction regions is given in the next section.

It will be more convenient for the subsequent analysis by having Eqs. (19a), (19b), and (22) in dimensionless forms and by integrating directly each of them. The equations then become, respectively,

$$\Delta^2 = -(\gamma - 1)^2 \left(\frac{p_1}{p_\infty}\right)^{-2} \left(\frac{I}{F}\right)^2 \left\{\exp\left[\int \frac{I(\Phi + 6) - h}{F(\Phi + 6)} \frac{d}{d\bar{\chi}} \ln\left(\frac{p_1}{p_\infty}\right)^{(\gamma-1)/\gamma} d\bar{\chi}\right]\right\} \times$$
$$\int \left\{\exp\left[\int \frac{I(\Phi + 6) - h}{F(\Phi + 6)} \frac{d}{d\bar{\chi}} \ln\left(\frac{p_1}{p_\infty}\right)^{-(\gamma-1)/\gamma} d\bar{\chi}\right]\right\} AF\left(\frac{p_1}{p_\infty}\right) \bar{\chi}^{-3} d\bar{\chi} \quad (24a)$$

$$\Delta^2 = -(\gamma - 1)^2 \left(1 - h\frac{T_c}{T_0}\right) \left(\frac{p_1}{p_\infty}\right)^{-2} \times \left(\frac{I}{J}\right)^2 \int \Phi \left(\frac{p_1}{p_\infty}\right) J \bar{\chi}^{-3} d\bar{\chi} \quad (24b)$$

$$\Delta^2 = -2\left[\frac{2}{\gamma(\gamma + 1)}\right]^{1/2} \int \left(\frac{p_1}{p_\infty} - 1\right) \left(\frac{p_1}{p_\infty} + \frac{\gamma - 1}{\gamma + 1}\right)^{-1/2} \bar{\chi}^{-3} d\bar{\chi} \quad (24c)$$

where $\bar{\chi}$ is the dimensionless form of the independent variable, x, and is called the interaction parameter,

$$\bar{\chi} \equiv \left(\frac{C_\infty \nu_\infty}{u_\infty}\right)^{1/2} M_\infty{}^3 x^{-1/2} \equiv M_\infty{}^3 \left(\frac{C_\infty}{Re_{x\infty}}\right)^{1/2} \quad (25a)$$

and Δ is the dimensionless boundary-layer thickness,

$$\Delta \equiv (u_\infty / C_\infty \nu_\infty M_\infty{}^5)(I\delta/h) \quad (25b)$$

The constants of integration in Eqs. (24a)–(24c) have been set to be zero, since the boundary-layer thickness is zero at the leading edge.

Approximate Solution

The virtue of the present integral approach, as also demonstrated in Refs. 3, 4, and 6, owes much to the fact that simple approximate expressions can be obtained for the complicated functions, such as F, J, I, and A, in the integrated equations. The Pohlhausen parameter Λ has been shown to be of the order $0[10^{-1}]$[3, 4] and the injection parameter Φ of the order $0[1]$[4] as a consequence of the assumption (3). Consequently, for the sweat-cooling case of small injection ($0 < \Phi < 1.5$), the following simplifications can be made for F, J, I, and A:

$$F = 0.118 \quad (26a)$$

$$J = j_0 + j_1\Phi \quad (26b)$$

where

$$j_0 \equiv 0.243(1 - h)$$

$$j_1 \equiv -0.0123(1 - h) - 0.0627h[1 - (T_c/T_0)]$$

$$I = i_0 + i_1\Phi \quad (26c)$$

where

$$i_0 \equiv 0.5h - 0.0825$$

$$i_1 \equiv 0.0188 + 0.1h[1 - (T_c/T_0)]$$

and

$$A = 2 + (2/3)\Phi \quad (26d)$$

Higher order approximations than those given above have also been used for a check of the degree of improvement in the final results. The resulted complicated algebraic manipulations, however, are not justified in view of the small differences found in the final results.

With the approximations (26a)–(26d), asymptotic solutions can be obtained in both the strong and the weak pressure interaction regions.

Weak Pressure Interaction Region ($\bar{\chi} \ll 1$)

The induced surface pressure in the weak interaction region can be expressed as:

$$(p_1/p_\infty) = 1 + p^{(1)}\bar{\chi} + 0[\bar{\chi}^2] \quad (27a)$$

where $p^{(1)}$ is a constant to be determined. The zero-order term is unity since the flow deflection due to the boundary layer is extremely small in the region far from the leading edge of the flat plate.

With the above expression for the induced pressure, Eqs. (24a)–(24c) require, in order that they be consistent among one another, the injection parameter being of the form

$$\Phi = \Phi^{(0)} + \Phi^{(1)}\bar{\chi} + 0[\bar{\chi}^2] \quad (27b)$$

where $\Phi^{(0)}$ and $\Phi^{(1)}$ are constants to be determined.

The functions J, I, and A, which were expressed in terms of Φ, can now be made in terms of the independent variable $\bar{\chi}$. They become

$$J = J^{(0)} + J^{(1)}\bar{\chi} + 0[\bar{\chi}^2] \quad (28a)$$

where

$$J^{(0)} \equiv j_0 + j_1\Phi^{(0)}$$

$$J^{(1)} \equiv j_1\Phi^{(1)}$$

$$I = I^{(0)} + I^{(1)}\bar{\chi} + 0[\bar{\chi}^2] \quad (28b)$$

where

$$I^{(0)} \equiv i_0 + i_1\Phi^{(0)}$$

$$I^{(1)} \equiv i_1\Phi^{(1)}$$

and

$$A = A^{(0)} + A^{(1)}\bar{\chi} + 0[\bar{\chi}^2] \quad (28c)$$

where

$$A^{(0)} \equiv 2 + (2/3)\Phi^{(0)}$$

$$A^{(1)} \equiv (2/3)\Phi^{(1)}$$

The dimensionless boundary-layer thickness, obtained by substituting Eq. (27a) into Eq. (24b), is

$$\Delta = \bar{\chi}^{-1}\{\Delta^{(0)} + \Delta^{(1)}\bar{\chi} + 0[\bar{\chi}^2]\} \tag{29}$$

where

$$\Delta^{(0)} \equiv (\gamma - 1)I^{(0)} \left[\frac{1}{2}\left(1 - h\frac{T_c}{T_0}\right)\frac{\Phi^{(0)}}{J^{(0)}}\right]^{1/2} \tag{30a}$$

and

$$\Delta^{(1)} \equiv \Delta^{(0)}\left[\frac{I^{(1)}}{I^{(0)}} + \frac{\Phi^{(1)}}{\Phi^{(0)}}\right] \tag{30b}$$

Substituting the above first-order expressions, Eqs. (27a)–(27b), Eqs. (28a)–(28c), and Eq. (29) into Eqs. (24a)–(24c) gives the solution as:

$$\Phi^{(0)} = -\frac{1}{2j_1}\left\{\left[j_0 + 3j_1 - \frac{3F}{2}\left(1 - h\frac{T_c}{T_0}\right)\right] + \left[\left(j_0 + 3j_1 - \frac{3F}{2}\left\{1 - h\frac{T_c}{T_0}\right\}\right)^2 - 12j_0j_1\right]^{1/2}\right\} \tag{31a}$$

$$\Phi^{(1)} = \frac{(\gamma - 1)}{2\gamma F}\, p^{(1)}A^{(0)}\left[\left(i_0 - \frac{h}{6}\right) + \left(i_1 + \frac{h}{36}\right)\Phi^{(0)}\right] \times \left[\left(\frac{A^{(0)}i_1}{I^{(0)}} + \frac{2}{3}\right) - \frac{F}{J^{(0)}}\left(1 - h\frac{T_c}{T_0}\right)\left(\frac{\Phi^{(0)}i_1}{I^{(0)}} + 1\right)\right]^{-1} \tag{31b}$$

and

$$p^{(1)} = (1/2)\gamma(\gamma - 1)I^{(0)}(A^{(0)}/2F)^{1/2} \tag{31c}$$

The asymptotic solution of (p_1/p_∞), Φ, and Δ can then be used to evaluate the asymptotic behaviors of the skin-friction coefficient and the Nusselt number in the weak interaction region. By definition, the skin-friction coefficient can be written in terms of the interaction parameter and the dimensionless boundary-layer thickness as

$$C_f \equiv \frac{2}{\rho_\infty u_\infty^2}\left(\mu\frac{\partial u}{\partial y}\right)_0 = \frac{(\gamma - 1)I}{M_\infty^3(\Phi + 6)\Delta} \times \left[12 + \frac{h\Delta^2\bar{\chi}^3}{\gamma(\gamma - 1)I^2}\frac{d}{d\bar{\chi}}\left(\frac{p_1}{p_\infty}\right)\right] \tag{32}$$

In the weak interaction region this becomes,

$$C_f = M_\infty^{-3}\bar{\chi}\{C_f^{(0)} + C_f^{(1)}\bar{\chi} + 0[\bar{\chi}^2]\} \tag{33}$$

where

$$C_f^{(0)} \equiv [12(\gamma - 1)I^{(0)}]/[\Delta^{(0)}(\Phi^{(0)} + 6)] \tag{34a}$$

and

$$C_f^{(1)} \equiv C_f^{(0)}\left[\frac{I^{(1)}}{I^{(0)}} - \frac{\Delta^{(1)}}{\Delta^{(0)}} - \frac{\Phi^{(1)}}{\Phi^{(0)} + 6} + \frac{hp^{(1)}}{12\gamma(\gamma - 1)}\left(\frac{\Delta^{(0)}}{I^{(0)}}\right)^2\right] \tag{34b}$$

The Nusselt number, by definition, is

$$Nu \equiv \frac{k_0(\partial T/\partial y)_0 x}{k_\infty(T_e - T_0)} = M_\infty\bar{\chi}^{-2}\left[\frac{I\Phi}{\Delta}\frac{[1 - (T_c/T_0)]}{(1 - h)}\right] \tag{35}$$

which, in the weak interaction region, gives,

$$Nu = M_\infty\bar{\chi}^{-1}\{Nu^{(0)} + Nu^{(1)}\bar{\chi} + 0[\bar{\chi}^2]\} \tag{36}$$

where

$$Nu^{(0)} \equiv \frac{I^{(0)}\Phi^{(0)}}{\Delta^{(0)}}\frac{[1 - (T_c/T_0)]}{(1 - h)} \tag{37a}$$

and

$$Nu^{(1)} \equiv 0 \tag{37b}$$

The vanishing first-order contribution in Nusselt number in the weak interaction region is not a complete surprise as will be discussed later.

Strong Pressure Interaction Region ($\bar{\chi} \gg 1$)

In the strong interaction region it has been shown[8] that the existence of a similar solution in the boundary layer requires the induced surface pressure to be directly proportional to the interaction parameter. Thus the expression for the induced pressure is given as,

$$(p_1/p_\infty) = \bar{\chi}\{p^{(0)} + p^{(1)}\bar{\chi}^{-1} + 0[\bar{\chi}^{-2}]\} \tag{38a}$$

where $p^{(0)}$ and $p^{(1)}$ are to be determined.

The injection parameter must have the form

$$\Phi = \Phi^{(0)} + \Phi^{(1)}\bar{\chi}^{-1} + 0[\bar{\chi}^{-2}] \tag{38b}$$

in order that Eqs. (24a)–(24c) are consistent among one another.

The approximate expressions for J, I, and A, Eqs. (26b)–(26d), thus become:

$$J = J^{(0)} + J^{(1)}\bar{\chi}^{-1} + 0[\bar{\chi}^{-2}] \tag{39a}$$

$$I = I^{(0)} + I^{(1)}\bar{\chi}^{-1} + 0[\bar{\chi}^{-2}] \tag{39b}$$

and

$$A = A^{(0)} + A^{(1)}\bar{\chi}^{-1} + 0[\bar{\chi}^{-2}] \tag{39c}$$

where the respective coefficients have the same expressions as those in the case of weak interactions.

The dimensionless boundary-layer thickness can be obtained by substitution of Eq. (38a) into Eq. (24c) as:

TABLE 1. Numerical Results for the Weak and the Strong Interactions

	$\bar{\chi} \ll 1$			$\bar{\chi} \gg 1$		
T_c/T_0	0.2	0.2	0.2	0.2	0.2	0.2
h	0.5	0.7	0.9	0.5	0.7	0.9
$\Phi^{(0)}$	1.985	0.941	0.216	1.441	0.715	0.207
$\Phi^{(1)}$	−0.161	−0.090	−0.035	0.572	0.262	0.075
$\Delta^{(0)}$	0.440	0.450	0.386	0.654	0.699	0.736
$\Delta^{(1)}$	−0.050	−0.050	−0.066	−0.696	−0.639	−0.611
$I^{(0)}$	0.285	0.338	0.387	0.252	0.321	0.386
$I^{(1)}$	−0.010	−0.006	−0.003	0.034	0.020	0.007
$p^{(0)}$	1.000	1.000	1.000	0.404	0.461	0.510
$p^{(1)}$	0.299	0.315	0.327	0.733	0.761	0.755
$C_f^{(0)}$	0.389	0.520	0.775	0.297	0.402	0.609
$C_f^{(1)}$	0.060	0.084	0.164	0.212	0.231	0.308
$Nu^{(0)}$	2.060	1.885	1.732	0.890	0.875	0.870
$Nu^{(1)}$	0.000	0.000	0.000	1.420	1.175	1.060

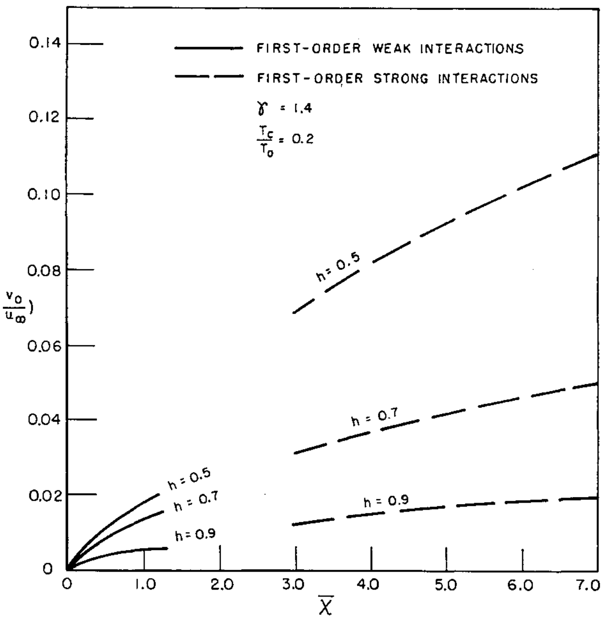

FIG. 2. Asymptotic behaviors of the coolant injection velocity in the weak and the strong interaction regions.

$$\Delta = \bar{\chi}^{-3/2}\{\Delta^{(0)} + \Delta^{(1)}\bar{\chi}^{-1} + 0[\bar{\chi}^{-2}]\} \quad (40)$$

where

$$\Delta^{(0)} \equiv \left[\frac{32p^{(0)}}{9\gamma(\gamma+1)}\right]^{1/2} \quad (41a)$$

and

$$\Delta^{(1)} = \frac{3}{10}\frac{\Delta^{(0)}}{p^{(0)}}\left[p^{(1)} - \frac{3\gamma+1}{\gamma+1}\right] \quad (41b)$$

With the above first-order expressions, Eqs. (24a)–(24c) are solved simultaneously to the first-order interactions, and the coefficients $\Phi^{(0)}$ and $p^{(0)}$ are given as

$$\Phi^{(0)} = (1/2R)\{-S + [S^2 + 4j_0R]^{1/2}\} \quad (42a)$$

where

$$R \equiv \left[\frac{\gamma-1}{2\gamma}\left(1 - h\frac{T_c}{T_0}\right)\left(i_1 + \frac{h}{36}\right) - \frac{1}{3}j_1\right]$$

$$S \equiv \left[\frac{\gamma-1}{2\gamma}\left(1 - h\frac{T_c}{T_0}\right)\left(i_0 - \frac{h}{6}\right) + \frac{F}{2}\left(1 - h\frac{T_c}{T_0}\right) - \frac{1}{3}j_0 - j_1\right]$$

and

$$p^{(0)} \equiv \left[\frac{9}{32}\gamma(\gamma+1)(\gamma-1)^2\left(1 - h\frac{T_c}{T_0}\right) \times \frac{(I^{(0)})^2\Phi^{(0)}}{J^{(0)}}\right]^{1/2} \quad (42b)$$

The general expressions for $\Phi^{(1)}$ and $p^{(1)}$ are extremely lengthy and are not given here. However, numerical values of $\Phi^{(1)}$ and $p^{(1)}$ for specific cases can be obtained with relative ease by substituting numerical constants into Eqs. (24a)–(24c) and solving these equations simultaneously. Values of $\Phi^{(1)}$ and $p^{(1)}$ for a number of specific cases are tabulated in the next section.

Consequently, the skin-friction coefficient in the strong interaction region, from Eq. (32), becomes

$$C_f = M_\infty^{-3}\bar{\chi}^{3/2}\{C_f^{(0)} + C_f^{(1)}\bar{\chi}^{-1} + 0[\bar{\chi}^{-2}]\} \quad (43)$$

where

$$C_f^{(0)} \equiv \left[\frac{(\gamma-1)I^{(0)}}{(\Phi^{(0)}+6)\Delta^{(0)}}\right]\left[12 + \frac{p^{(0)}h}{\gamma(\gamma-1)}\left(\frac{\Delta^{(0)}}{I^{(0)}}\right)^2\right] \quad (44a)$$

and

$$C_f^{(1)} \equiv C_f^{(0)}\left[\frac{I^{(1)}}{I^{(0)}} - \frac{\Delta^{(1)}}{\Delta^{(0)}} - \frac{\Phi^{(1)}}{\Phi^{(0)}+6}\right] + \left[\frac{(\gamma-1)I^{(0)}}{(\Phi^{(0)}+6)\Delta^{(0)}}\right]\left[\frac{2p^{(0)}h}{\gamma(\gamma-1)}\left(\frac{\Delta^{(0)}}{I^{(0)}}\right)^2\right] \times \left[\frac{\Delta^{(1)}}{\Delta^{(0)}} - \frac{I^{(1)}}{I^{(0)}}\right] \quad (44b)$$

The Nusselt number, from Eq. (35), becomes

$$Nu = M_\infty\bar{\chi}^{-1/2}\{Nu^{(0)} + Nu^{(1)}\bar{\chi}^{-1} + 0[\bar{\chi}^{-2}]\} \quad (45)$$

where

$$Nu^{(0)} \equiv \frac{I^{(0)}\Phi^{(0)}}{\Delta^{(0)}}\frac{\left(1 - \frac{T_c}{T_0}\right)}{(1-h)} \quad (46a)$$

and

$$Nu^{(1)} \equiv Nu^{(0)}\left[\frac{I^{(1)}}{I^{(0)}} + \frac{\Phi^{(1)}}{\Phi^{(0)}} - \frac{\Delta^{(1)}}{\Delta^{(0)}}\right] \quad (46b)$$

The above formulas for the injection parameter, the induced surface pressure, the dimensionless boundary-

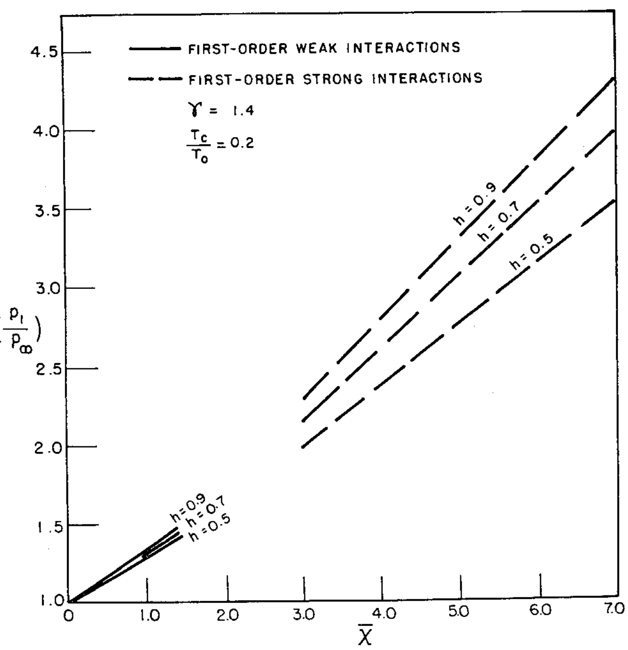

FIG. 3. Asymptotic behaviors of the induced surface pressure in the weak and the strong interaction regions.

layer thickness, the skin-friction coefficient, and the Nusselt number will be used in the following section to obtain numerical results for certain cases in both the strong and the weak interaction regions.

Numerical Results and Discussions

The distribution of coolant injection velocity is not explicitly expressed in the above solution, but it can be obtained without much difficulty from the definition of the injection parameter, Eq. (10e). It is given as

$$\left(\frac{v_0}{u_\infty}\right) = \frac{h}{M_\infty}\left(\frac{\gamma-1}{2}\right)^2\left(\frac{p_i}{p_\infty}\right)^{-1}\left(\frac{\Phi I}{\Delta}\right) \tag{47}$$

which becomes, in the weak interaction region,

$$\left(\frac{v_0}{u_\infty}\right) = \frac{h}{M_\infty}\left(\frac{\gamma-1}{2}\right)^2\frac{\Phi^{(0)}I^{(0)}}{\Delta^{(0)}}\,\bar{\chi}\times \{1 - p^{(1)}\bar{\chi} + 0[\bar{\chi}^2]\} \tag{48a}$$

and, in the strong interaction region,

$$\left(\frac{v_0}{u_\infty}\right) = \frac{h}{M_\infty}\left(\frac{\gamma-1}{2}\right)^2\frac{\Phi^{(0)}I^{(0)}}{p^{(0)}\Delta^{(0)}}\,\bar{\chi}^{1/2}\left\{1 + \left(\frac{\Phi^{(1)}}{\Phi^{(0)}} + \frac{I^{(1)}}{I^{(0)}} - \frac{\Delta^{(1)}}{\Delta^{(0)}} - \frac{p^{(1)}}{p^{(0)}}\right)\bar{\chi}^{-1} + 0[\bar{\chi}^{-2}]\right\} \tag{48b}$$

Shown in Fig. 1 is a representation of the coolant injection distribution along the surface of the porous flat plate.

Three specific cases have been computed and their results are presented in Table 1 and in Figs. 2 to 5. These cases are for $\gamma = 1.4$ and $(T_c/T_0) = 0.2$, but with different values of h, namely, $h = 0.5$, 0.7, and 0.9. The ratio of (T_c/T_0), chosen as 0.2, is based on the practical consideration of having a coolant temperature in the range from 500–600°R and a limiting surface temperature from 2,500–3,000°R. The three values of h cover approximately the range of free-stream Mach number from 6–9, as indicated in Eq. (15b). This range is usually regarded as such that the hypersonic behavior exists for flow over slender bodies.

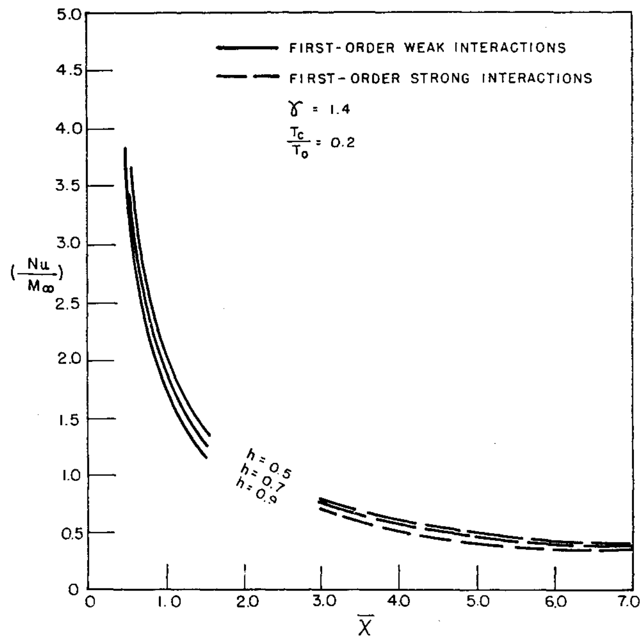

FIG. 5. Asymptotic behaviors of the Nusselt number in the weak and the strong regions.

The numerical results in Table 1 can easily be used to verify that the assumption of the Pohlhausen parameter being of the order $0[10^{-1}]$ is a valid one for the sweat-cooled case as well as for the insulated case.[4] The results also indicate that the restriction for the range of the injection parameter $(0 < \Phi < 1.5)$ is quite reasonable except in the weak interaction region for the case $(T_c/T_0 = 0.2, h = 0.5)$. This fact explains why the trends of the weak and the strong interaction solutions, as shown in Fig. 2, seem to match quite well except in the case $(T_c/T_0 = 0.2, h = 0.5)$. In the three cases under consideration the coolant injection velocity at the surface is found in Fig. 2 to vary from zero to ten percent of the free-stream sonic velocity (u_∞/M_∞). This again justifies the assumption in Eq. (3), $v_0^2 \ll u_\infty^2$.

The effects of sweat-cooling on the hypersonic viscous behaviors are shown in Figs. 3 to 5. The induced surface pressure decreases as the plate temperature decreases. The skin-friction coefficient, however, exhibits a more pronounced effect due to sweat-cooling. The variation of the Nusselt number along the plate surface remains almost the same for all three cases. This does not mean that the same amount of cooling occurs in the three cases since the plate temperatures are different. In the weak interaction region, it is indicated in Eq. (37b) that the first-order coefficient in the asymptotic expression of the Nusselt number or the local heat transfer coefficient vanishes. This agrees with the conclusion reached by Maslen[9] for an

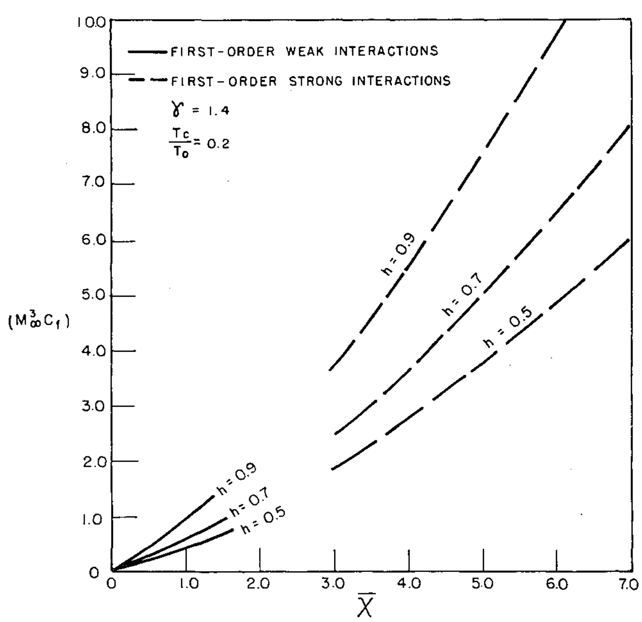

FIG. 4. Asymptotic behaviors of the skin-friction coefficient in the weak and the strong interaction regions.

impermeable surface, that the local heat transfer rate is unaltered to first order by the self induced pressure gradient. Thus the present analysis shows that the same statement is valid in the weak interaction region over a surface with mass transfer.

The direct application of the above results to actual physical systems is quite limited in view of the many assumptions and approximations being made in the present analysis. Among them the nonuniform coolant injection velocity as required by the similarity consideration seems to be most restrictive. A nonsimilar solution such as for a uniform coolant injection would be possible by application of some numerical scheme, such as the technique used by Pallone.[10] The present analysis, however, does present a comparatively simple approach and gives an analytical form of the asymptotic behaviors. The results obtained may be used as a preliminary estimate to the actual complicated system.

References

[1] Yasuhara, M., *On the Hypersonic Viscous Flow Past a Flat Plate With Suction or Injection*, J. of Physical Soc. of Japan, Vol. 12, pp. 177–182, 1957.

[2] Li, T. Y., and Gross, J. F., *Hypersonic Strong Viscous Interaction on a Flat Plate With Surface Mass Transfer*, 1961 Heat Transfer and Fluid Mechanic Institute, pp. 146–160.

[3] Nagakura, T., and Naruse, H., *An Approximate Solution of Hypersonic Laminar Boundary-Layer Equations and Its Application*, J. of Physical Soc. of Japan, Vol. 12, pp. 1298–1304, 1957.

[4] Tien, C. L., *On Hypersonic Viscous Flow Over an Insulated Flat Plate with Surface Mass Transfer*, Journal of Aerospace Sciences, Vol. 29, No. 9, Sept., 1962.

[5] Morduchow, M., *On Heat Transfer Over a Sweat-Cooled Surface in Laminar Compressible Flow with a Pressure Gradient*, Journal of the Aeronautical Sciences, Vol. 19, No. 10, Oct. 1952.

[6] Naruse, H., *Approximate Solutions of the Hypersonic Laminar Boundary Layer Equations With Heat Transfer and Arbitrary Pressure Gradient and Their Applications*, Rep. No. 361, Aero. Res. Inst., Univ. of Tokyo, Japan, 1961.

[7] Libby, P. A., and Morduchow, M., *Method For Calculation of Compressible Laminar Boundary Layer with Axial Pressure Gradient and Heat Transfer*, NACA TN 3157, 1954.

[8] Hayes, W. D. and Probstein, R. F., *Hypersonic Flow Theory*, Chap. 9, Academic Press, New York, 1959.

[9] Maslen, S. H., *Second Approximation to Laminar Compressible Boundary Layer on Flat Plate in Slip Flow*, NACA TN 2818, 1952.

[10] Pallone, A., *Nonsimilar Solutions of the Compressible-Laminar Boundary-Layer Equations With Applications to the Upstream-Transpiration Cooling Problem*, Journal of the Aerospace Sciences, Vol. 28, No. 6, June 1961.